Spectral Properties of Band Irreducible Operators

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Abstract. Number of spectral properties of a band irreducible operator $T$ on a Banach lattice $E$ will be discussed. If $T$ is $\sigma$-order continuous, $r(T)$ is a pole of the resolvent $R(\cdot, T)$, and the band $E^\sim_c$ of all $\sigma$-order continuous functionals on $E$ is nonzero, then we prove among others that $r(T) > 0$, that $T$ has an eigenvector which is a weak unit, and that the adjoint $T^*$ of $T$ has a positive order continuous eigenvector. Furthermore, we provide some criteria of primitivity for band irreducible operators in terms of limits of real sequences. Finally, we discuss the question whether the operator inequalities $0 \leq S < T$ imply the spectral radius inequality $r(S) < r(T)$, where $T$ is a band irreducible operator on $E$.

Key words: Banach lattice, Banach function space, positive operator, integral operator, spectrum, primitivity, Hermitian operator

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Spectral theory on positive operators occupies a major place in the general theory of operators on Banach lattices. Over the last decades, special attention has been paid to spectral properties of ideal and band irreducible operators as a natural extension of the well-known concept of irreducible matrices by Perron-Frobenius (see [1], Chapter 8 or [15], Chapter 1). This study has been really launched thanks to Andô and Krieger ([1], p. 431), whom investigated the spectral radius of some integral operators on Dedekind complete Banach lattices. Since then, Schaefer, Sawashima, Niiro, Stecenko, and de Pagter (see [15, 1, 11] for precise reference) contributed in the building of the theory of ideal irreducible operators and answered most questions and problems in this direction. Later, Grobler, Schaefer, Caselles, Abramovich, Aliprantis, and Burkinshaw (see [1, 11] for precise reference) studied the band irreducible operators on Banach lattices and showed that this theory also deserves a special consideration. In this prospect, the main purpose has been to study the extent to which the classical Andô-Krieger theorem can be generalized. However, along these lines several other aspects of band irreducible operators have receive almost no attentions. This note tries to make some contribution in this direction. A short synopsis of the content of this note seems to be in order.

In the first section, there are conditions of existence of eigenvector and eigenfunctional under assumptions that are much more general than compactness. In the second section, we provide criteria of primitivity and imprimitivity for band irreducible operators, which are analogous to famous criteria from the theory of irreducible matrices. In the final section we focus on the question whether the operator inequalities $0 \leq S < T$ imply the spectral radius inequality $r(S) < r(T)$, where $T$ is a band irreducible operator on a Banach lattice $E$.

For terminology, notions, and properties on the theory of Banach lattices not explained or proved in this note, we refer to [1]; see also [8, 15].
1 Theorem about existence of eigenvector and eigenfunctional

Recall ([1], p. 349) that a positive operator $T$ on a Banach lattice $E$ is *ideal irreducible* if $T$ has no invariant non-trivial closed ideals and *band irreducible* if $T$ has no invariant non-trivial bands.

In the following theorem main spectral properties of band irreducible operators are added.  

**Theorem 1.** Let $E$ be a Banach lattice, the dimension of $E$ is at least two and the σ-order continuous dual $E^*_c \neq \{0\}$. Suppose that $T$ is a band irreducible σ-order continuous operator on $E$ such that the point $\lambda_0 = r(T)$ is a pole of the resolvent $R(\cdot, T)$. Then:

(a) the spectral radius $r(T) > 0$;

(b) the eigenvector $x_0$ corresponding to the eigenvalue $r(T)$ of the operator $T$ is a weak unit and $\dim N(r(T)I - T) = 1$;

(c) the eigenfunctional $x_0^*$ corresponding to the eigenvalue $r(T)$ of the operator $T^*$ is a strictly positive order continuous functional and $\dim N(r(T)I - T^*) = 1$;

(d) $\lambda_0 = r(T)$ is a simply pole of $R(\cdot, T)$;

(e) the residuum of $R(\cdot, T)$ at $\lambda_0 = r(T)$ is a rank-one operator.

**Proof.** Observe first that the condition $E^*_c \neq \{0\}$ and the existence on $E$ of a σ-order continuous band irreducible operator imply [16] that the band $E^*_c$ separates the points of $E$. Thus a Riesz dual system $\langle E, E^*_c \rangle$ is defined and moreover there exists a strictly positive functional on $E$. Therefore $E$ has the countable sup property, hence the band $E^n_\sigma$ of order continuous functionals is equal to the band $E^*_c$ σ-order continuous functionals.

The eigenvector $x_0 > 0$ corresponding to the eigenvalue $r(T)$ of the operator $T$ exists ([15], p. 352): $Tx_0 = r(T)x_0$. The band irreducibility of the operator $T$ implies $r(T) > 0$ hence the assertion (a) holds and $x_0$ is a weak unit.

It is no loss of generality to assume that $r(T) = 1$. The band $E^n_\sigma$ is invariant under $T^*$, let $T'$ be a restriction of $T^*$ on $E^n_\sigma$. The operator $T'$ is band irreducible. Indeed, assume by the way of a contradiction that there exists a non-trivial band $B$ invariant under $T'$, $\{0\} \subset B \subset E^n_\sigma$. The band $B$ is $\sigma(E^n_\sigma, E)$-closed. It is by the equality [10] $B^o \cap E^n_\sigma = B$, where the polar is taken in the dual system $\langle E, E^\sigma \rangle$. The polar $B^o$ of the band $B$ with respect to the dual system $\langle E, E^\sigma \rangle$ is an ideal invariant under the operator $T$. If $x \in E$ and there exists a net of elements $x_\alpha$ from $B^o$ such that $0 \leq x_\alpha \uparrow |x|$, then $x^*|x| = \lim x^*x_\alpha = 0$ for every functional $x^* \in B$, hence $x \in B^o$, therefore $B^o$ is a band. By bipolar theorem ([8], p. 140) the band $B^o$ can’t be trivial and we get a contradiction with the band irreducibility of $T$.

For an arbitrary functional $x^* \in E^n_\sigma$ we get $x^*x_0 = x^*(Tx_0) = (T'x^*)x_0$; it follows that $((I - T')x^*)x_0 = 0$ and therefore $\lambda_0 = 1$ is a point of the spectrum of the operator $T'$. Since the inequality $r(T') \leq 1$ holds, we get $r(T') = 1$. The inclusion ([1], p. 256) $\rho_\infty(T^*) \subset \rho(T')$, where $\rho_\infty(T^*)$ is the unbounded connected component of the resolvent set of the operator $T^*$, implies that the number $\lambda_0 = 1$ is an isolated point of the spectrum $\sigma(T')$ of the operator $T'$. For all $\lambda$ sufficiently close to point $\lambda_0 = 1$, the inequality

$$|\lambda - 1|^{k+1} |R(\lambda, T')| \leq |\lambda - 1|^{k+1} |R(\lambda, T^*)|$$

holds, where $k$ is the order of the pole $\lambda_0 = 1$ of the resolvent of the operator $T$. It follows that

$$\lim_{\lambda \to 1} (\lambda - 1)^{k+1} R(\lambda, T') = 0.$$
Hence the point $\lambda_0 = 1$ is a pole of the resolvent of the operator $T'$. This shows that there exists a strictly positive functional $x_0^* \in E_\sim^*$ such that $T'x_0^* = x_0^*$.

Now assume that there exists an element $x \in E$, which is an eigenvector of $T$ corresponding to the eigenvalue $\lambda_0 = 1$: $Tx = x$. Then $T|x| = |x|$. Adding this equality with the equality $Tx = x$ we get $Tx^+ = x^+$ and $Tx^- = x^-$. The band irreducibility of $T$ implies either $x^+ = 0$ or $x^- = 0$. In other words, every element from the eigenspace of the operator $T$ corresponding to $\lambda_0 = 1$ belongs either $E^+$ or $-E^+$. This shows ([15], p. 66) $\dim(N(I - T)) = 1$ and the assertion (b) is proved.

If $k$ is the order of a pole of $R(., T)$ around $\lambda_0 = 1$, then the coefficient of the Laurent series expansion of $R(., T)$ around $\lambda_0 = 1$ by $(\lambda - \lambda_0)^{-k}$ presents a positive operator which satisfies the identity ([1], p. 265) $T_{-k} = (T - I)^{k-1}T_{-1}$, where $T_{-1}$ is a residuum of the function $R(., T)$ at the point $\lambda_0 = 1$. Then by $k > 1$ the equalities

$$x_0^*(T_{-k}x) = x_0^*((T - I)^{k-1}T_{-1}x) = 0$$

hold. Hence $T_{-k}x = 0$ for all $x \geq 0$ as $x_0^*$ is strictly positive. Whence $T_{-k} = 0$, which is a contradiction. Thus $k = 1$ and the assertion (e) follows. Next, the equalities ([1], p. 74, 268) $\dim(N(I - T)) = \dim(N(I - T^*))$ and $R(T_{-1}) = N(I - T)$ give assertions (c) and (d). □

For the case of ideal irreducible operators, the result analogous to theorem 1 was established by Sawashima [14]. Note that the existence on a given Banach lattice $E$ of an ideal irreducible operator is the stronger assumption than the assumption about the existence of a band irreducible operator. For example, every Banach function space $X$ admits the band irreducible integral operator, while an ideal irreducible operator on $X$ may not exist; see [6, 7] for details.

The existence of an eigenvector at a compact $\sigma$-order continuous band irreducible operator (by condition $E_\sim^* \neq \{0\}$) was first noted in [16]. For the case of a band irreducible integral operator on a Banach function space the analog of theorem 1 was obtained in [7].

## 2 Primitivity and imprimitivity

As for nonnegative matrices, a band irreducible operator $T$ with $r(T) > 0$ is called **imprimitive**, if the peripheral spectrum of $T$ consists of more than one point, and **primitive**, if it consists of only one point $r(T)$.

Let $T$ be a band irreducible integral operator on a Banach function space $X$ on a set $\Omega$ with a measure $\mu$,

$$Tx(t) = \int_{\Omega} k(t,s)x(s) \, d\mu(s).$$

Recall that by theorems of Lozanovsky and Andö-Krieger ([1], p. 199, 369) $r(T) > 0$.

Next result [7] is an analog of the classical Frobenius theorem about a general form of imprimitive matrices ([15], p. 22-23).

**Theorem 2.** Let $T$ be a band irreducible integral operator with kernel $k(t,s)$ on a Banach function space $X$ such that the point $\lambda_0 = r(T)$ is a pole of resolvent $R(., T)$. Then the peripheral spectrum $\lambda$ of the operator $T$ has the form $r(T)H_m$, where $H_m$ is the group of all $m$-th roots of unity. In particular the spectrum $\sigma(T)$ of the operator $T$ is invariant under the
rotation on angle $\frac{2\pi}{m}$ and \dim N(\lambda I - T) = \dim N(\lambda I - T^*) = 1$ for all $\lambda \in r(T)H_m$. In case $m > 1$ there exists a partition of the set $\Omega$ on $m$ disjoint sets $\Omega_j$ with positive measure, $\Omega = \bigcup_{j=1}^{m} \Omega_j$, such that $k(t, s) = 0$ for $t \in \Omega_j$, $s \notin \Omega_{j+1}$ (in case $j = m$ by definition $j+1 := 1$).

On the other hand, if for some $m > 1$ a partition of $\Omega$ mentioned above exists, then $H_m \subseteq H$.

**Proof.** We can suppose $r(T) = 1$. From theorem 1 and Lotz-Schaefer theorem ([15], p. 331) we infer that $H$ consists entirely of finite number of poles. By $x_0$ and $x^*_0$ we denote a positive function and a strictly positive functional (that exist again by theorem 1) such that $Tx_0 = x_0$, $T^*x^*_0 = x^*_0$.

Let us verify that $H$ is the group of all $m$-th roots of unity by some $m$. We consider the ideal $I_{x_0}$ generated by the element $x_0$. Then $T(I_{x_0}) \subseteq I_{x_0}$. The ideal $I_{x_0}$, equipped with the norm $\|x\|_{x_0} = \inf \{c : |x| \leq cx_0\}$, is a $M$-space with the unit $x_0$. By Kakutani-Bohnenblust-Krein theorem ([8], p. 194-195 or [1], p. 95) the space $I_{x_0}$ is lattice isometric to the space of continuous functions $C(K)$ on a compact Hausdorff space $K$, $I_{x_0} \ni x \leftrightarrow \hat{x} \in C(K)$.

Moreover, we can assume that $\hat{x}_0 = 1$. The operator $\hat{T}\hat{x} := \hat{T}x$ on $C(K)$ is continuous. Clearly, $r(\hat{T}) = 1$. Fix $\lambda \in H$. Number $\lambda$ is an eigenvalue of the ideal $\hat{T}$. Then ([12], lemma 5.1(H)) there exists an operator $S$ on $C(K)$ such that $\hat{T} = \lambda^{-1} S^{-1} \hat{T} S$. Now if $\hat{T}x_1 = \lambda_1 x_1$, $\lambda_1 \in H$, then $\lambda x_1 = \lambda^{-1} S^{-1} \hat{T} S x_1$, hence $\lambda_1 x_1 = \hat{T} S x_1$, therefore $\lambda_1$ is an eigenvalue of the operator $\hat{T}$ and it follows that $\lambda_1 \in H$. Thus $\lambda H \subseteq H$ for all $\lambda \in H$. It implies that $H$ is the group of $m$-th roots of unity, that is $H = H_m$.

Let $x_1$ and $x_2$ be eigenvectors of the operator $T$ corresponding to some $\lambda \in H$, that is $Tx_i = \lambda x_i$, and moreover $\|x_i\| = 1$, $i = 1, 2$. Then $\hat{x}_i \in C(K)$. Since $\frac{1}{\lambda} \in H$, there exists an operator $S$ such that $\hat{T} = \lambda S^{-1} \hat{T} S$. Equalities $S\hat{x}_1 = \hat{T} S x_1$, $i = 1, 2$, hold, hence $S\hat{x}_1 = S\hat{x}_2$ or $S\hat{x}_1 = -S\hat{x}_2$, therefore $\hat{x}_1 = \hat{x}_2$ or $\hat{x}_1 = -\hat{x}_2$. Thus $\dim N(\lambda I - T) = \dim N(\lambda I - T^*) = 1$.

Let the operator $T$ be imprimitive. Again using the restriction of $T$ on $I_{x_0}$ and lemma 5.1(II) from [12] we conclude that there exist functions $y_i \in X$ $(i = 1, \ldots, m)$ such that

$$y_i \wedge y_j = 0, \quad i \neq j, \quad T y_i+1 = y_i, \quad \sum_{i=1}^{m} y_i = x_0.$$ 

Let $\Omega_i = \text{Supp } y_i$. Then $\Omega_i \cap \Omega_j = \emptyset$, $i \neq j$, $\Omega = \bigcup_{i=1}^{m} \Omega_i$ and

$$T y_{i+1} = \int_{\Omega} k(t, s) y_{j+1}(s) \, d\mu(s) = \int_{\Omega_{j+1}} k(t, s) y_{j+1}(s) \, d\mu(s) = y_j = \chi_{\Omega_j} y_j,$$

that is $k(t, s) = 0$, if $t \notin \Omega_j$, $s \in \Omega_{j+1}$ for every $j = 1, \ldots, m$ or $k(t, s) = 0$, if $t \in \Omega_j$, $s \notin \Omega_{j+1}$ for $j = 1, \ldots, m$.

To proof the converse, suppose that there exists the partition, which is mentioned in the condition, of the set $\Omega$ on $m$ disjoint sets. Let us show that $H_m$ belongs to the peripheral spectrum. Let $\xi_k = \varepsilon^{-k} T^k P_{\Omega_0} x_0$, where $\varepsilon$ is an arbitrary $m$-th root of unity, $P_A$ defined by $P_A x = \chi_A x$ for arbitrary measurable set $A$ and $x \in X$. Then function $x_{\varepsilon} = \sum_{k=0}^{m-1} \xi_k$ is $\mu$-almost not vanishing and $T x_{\varepsilon} = \varepsilon x_{\varepsilon}$. Since $\varepsilon$ is an arbitrary $m$-th root of unity, we get that $H_m$ belongs to the peripheral spectrum $H$ of the operator $T$.

The proof will be finished if we show that all spectrum of the operator $T$ is invariant under the rotation on angle $\frac{2\pi}{m}$. Determine an operator $D$ on $X$ by an equality $D = \sum_{j=1}^{m} \varepsilon^{2 \pi i j / m} P_{\Omega_j}$. Then
D is invertible and it is easy to see the justice of the equality $T = e^{\frac{2\pi i}{m}}DTD^{-1}$. Therefore we get $\sigma(T) = \sigma(e^{\frac{2\pi i}{m}}DTD^{-1}) = e^{\frac{2\pi i}{m}}\sigma(T)$ and the proof is finished.

For the case of a compact operator $T$ analog of theorem 2 was mentioned in [17], p. 304.

In case of an arbitrary Banach lattice $E$ the next result common to theorem 2 holds. It’s proof is analogous and will be omitted.

**Theorem 2’.** Under assumptions of theorem 1 the peripheral spectrum $H$ of the operator $T$ has the form $r(T)H_m$. In particular $\dim N(\lambda I - T) = \dim N(\lambda I - T^*) = 1$ for all $\lambda \in r(T)H_m$ and there exist elements $y_i, i = 1, \ldots, m$ such that $y_i \wedge y_j = 0, i \neq j$, $Ty_{i+1} = y_i$ and $\sum_{i=1}^{m} y_i = x_0$. Moreover, if all $y_i$ are projection elements, then the spectrum $\sigma(T)$ is invariant under the rotation on angle $\frac{2\pi}{m}$. On the other hand, if for some $m > 1$ the collection of elements $y_i$ mentioned above exists, then $H_m \subseteq H$.

By the virtue of theorem 2 a nonnegative kernel $k(t, s)$ such that $k(t, s)$ defines the integral operator on some Banach function space $X$, is called primitive, if the partition of the set $\Omega$ mentioned in theorem 2 doesn’t exist for $k(t, s)$. A question arises naturally, namely, can we assert that some iterated kernel of a primitive kernel is positive ($\mu \times \mu$-almost)? Otherwise is it true for a primitive kernel an analog of the result about the positivity of some power of a primitive matrix? Let us show that it is not.

**Example 3.** Consider the infinite matrix $K = (k_{ij})_1^\infty$ such that elements $k_{ij}$ are nonnegative, $\sum_{i=1}^\infty \sum_{j=1}^\infty k_{ij}^2 < \infty$ and $k_{ij} = 0$ iff $j > 1, j \neq i + 1$, this matrix has the form

$$K = \begin{pmatrix}
  k_{11} & k_{12} & 0 & 0 & \ldots \\
  k_{21} & 0 & k_{23} & 0 & \ldots \\
  k_{31} & 0 & 0 & k_{34} & \ldots \\
  k_{41} & 0 & 0 & 0 & \ldots \\
  \ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}.$$ 

This matrix defines the ideal irreducible compact integral operator $T$ on $\ell_2$. The kernel of this operator, that is the matrix $K$, is primitive. Nevertheless, every power of the operator $T$ is not operator with positive kernel.

Common examples exist in every infinite dimensional Banach function space.

Nevertheless, the next result holds.

**Theorem 4.** Let $X$ be a Banach function space, $T$ on $X$ be a band irreducible integral operator with kernel $k(t, s)$. Then kernel $k(t, s)$ is primitive iff the equality

$$\lim_{n \to \infty} (\pi \times \pi)(\text{Supp } k^{(n)}(t, s)) = 1$$

holds, where $k^{(n)}(t, s)$ are iterated kernels of operators $T^n$ and a probability measure $\pi$ presents a norming of measure $\mu$, i.e. the relation $\pi(A) = \int_A \eta(s) \, d\mu(s)$ holds, the function $\eta(s)$ is $\mu$-almost positive and $\int_{\Omega} \eta(s) \, d\mu(s) = 1$.

**Proof.** The sufficiency is obvious, let us show the necessity. First of all, we can assume that $\mu$ is a probability measure. Indeed, consider an integral operator $T_x$ with the primitive kernel $k_x(t, s) = k(t, s)\eta(s)$ on a Banach function space $X$ with the measure $\pi$. Then the equality
\[ k^{(n)}_q(t, s) = k^{(n)}(t, s) \eta(s) \] implies that \( \text{Supp } k^{(n)}_q(t, s) = \text{Supp } k^{(n)}(t, s) \) on \( \Omega \times \Omega \). Thus we assume that \( \mu \) is a probability measure and moreover \( \pi = \mu \).

We shall verify first the assertion of the theorem in case when \( X \) presents the space \( L_\infty(\mu) \) of all bounded measurable functions, and the kernel \( k(t, s) \) of the operator \( T \) is bounded. It is no loss of generality to assume that \( r(T) = 1 \). The operator \( T \) is weak compact, hence, in virtue of Dunford-Pettis theorem ([8], p. 337), we conclude that the operator \( T^2 \) is compact. Therefore \( \lambda_0 = 1 \) is a pole of \( R(., T) \). Then theorem 1 and 2 imply that the operator \( T \) is primitive and the point \( \lambda_0 = 1 \) presents a simply pole of \( R(., T) \). Let an operator \( P \) be the residuum at this point. Then \( P \) is a positive projection onto the one-dimensional space \( N(I - T) \) of fixed points of the operator \( T \) and the equalities \( TP = PT = P \) hold. The projection \( P \) is presented in the integral form

\[
P x(t) = x_0(t) \int_\Omega x_0(s)x(s) \, d\mu(s),
\]

where \( \mu \)-almost positive functions \( x_0 \in L_\infty(\mu) \) and \( x_0' \in L_1(\mu) \) satisfy the equalities \( Tx_0 = x_0 \) and \( T^*x_0' = x_0' \). The operator \( T \) is presented in the form \( T = P + TQ \), where \( Q \) is defined by the equality \( Q = I - P \). Since ([1], p. 266) \( P \) is a spectral projection associated with the point \( \lambda_0 = 1 \) we infer that \( r(TQ|_{R(Q)}) < 1 \). Then the inequality

\[
\| (TQ)^n \| \leq \| (TQ|_{R(Q)})^n \| \| Q \|
\]
implies \( r(TQ) < 1 \). For powers of the operator \( T \) the equality \( T^n = P + T^nQ \) holds. Since \( r(TQ) < 1 \), then Gelfand formula ([1], p. 243) implies the convergence \( \| T^nQ \| \to 0 \) by \( n \to \infty \). Operators \( T^nQ \) are regular integral operators. Denote by \( d^{(n)}(t, s) \) kernels of \( T^nQ \). Relations

\[
\int_\Omega \int_\Omega |d^{(n)}(t, s)| \, d\mu(s) d\mu(t) \leq \| T^nQ \| \to 0
\]
imply that the sequence of functions \( d^{(n)}(t, s) \) converges in measure \( \mu \times \mu \) to zero. By virtue of the equality \( k^{(n)}(t, s) = x_0(t)x_0'(s) + d^{(n)}(t, s) \) we obtain the convergence of functions’ sequence \( k^{(n)}(t, s) \) in measure \( \mu \times \mu \) to the \( \mu \times \mu \)-almost positive on \( \Omega \times \Omega \) function \( x_0(t)x_0'(s) \). The desired assertion for operators on \( L_\infty(\mu) \) with bounded kernels is proved.

Now consider the general case of a Banach function space \( X \) and an integral operator \( T \). An operator \( R x(t) = \int_\Omega x(s) \, d\mu(s) \) on \( L_\infty(\mu) \) is integral with kernel \( r(t, s) = 1 \). Consider an operator \( B \) defined by a kernel \( b(t, s) = \min\{k(t, s), r(t, s)\} \). This operator acts on \( L_\infty(\mu) \) and, since \( \text{Supp } b(t, s) = \text{Supp } k(t, s) \), it is primitive. Then, as showed above, for kernels \( b^{(n)}(t, s) \) of operators \( B^n \) the equality \( \lim_{n \to \infty} (\mu \times \mu)(\text{Supp } b^{(n)}(t, s)) = 1 \) holds. Finally, notice that \( b^{(n)}(t, s) \leq k^{(n)}(t, s) \) and the proof is finished.

For the case of a separable measure \( \mu \) theorem 4 was obtained in [4]. Note that the sequence \( (\pi \times \pi)(\Omega_n) \) from theorem 4 doesn’t convergence “strictly monotonically” to zero even in a finite dimensional space, that is when the operator \( T \) defined by a matrix (see [5] for details).

In case of an arbitrary Banach lattice \( E \) the next theorem holds.

**Theorem 4’.** Under assumptions of theorem 1 next assertions are equivalent:

(a) the operator \( T \) is primitive;

(b) for every non-zero positive functional \( x^* \in E^* \) and element \( x > 0 \) \( \lim_{n \to \infty} x^*(T^n x) > 0 \);

(c) for every non-zero positive functional \( x^* \in E^*_e \) and element \( x > 0 \) \( \lim_{n \to \infty} \inf_{n \to \infty} x^*(T^n x) > 0 \).
3 When does \( 0 \leq S < T \) imply \( r(S) < r(T) \)?

It is well known that if for an irreducible matrix \( T \) the inequalities \( 0 \leq S < T \) hold, then \( r(S) < r(T) \). A question arises naturally, namely, when does the analogous result hold in an arbitrary Banach lattice \( E \)? As it will be shown below, the assumption that \( r(T) \) is a pole \( R(.,T) \) is sufficient and, even in some sense, necessary.

So, the next theorem holds.

**Theorem 5.** Under assumptions of theorem 1 the inequalities \( 0 \leq S < T \) imply the inequality \( r(S) < r(T) \).

**Proof.** Assume by the way of a contradiction that \( r(S) = r(T) > 0 \). Then [9] the point \( \lambda_0 = r(S) \) is a pole of \( R(.,S) \). Therefore

\[
 r(S)x = Sx \leq Tx
\]

for some \( x > 0 \). Let \( x_0^* \) be a strictly positive functional such that \( T^* x_0^* = r(T) x_0^* \). From relation (*) we infer

\[
 r(T)x_0^*x = r(S)x_0^*x \leq x_0^*(Tx) = r(T)x_0^*x,
\]

hence \( Tx = r(S)x = r(T)x \), therefore \( x \) is a weak unit. The equality \( Sx = Tx \) implies \( S = T \), and we get a contradiction. \( \square \)

The assumption that \( r(T) \) is a pole, is essential. Indeed, an ideal irreducible order continuous operator \( Q \) on the space \( L_1 \) exists with \( r(Q) = 0 \) ([15], p. 353). Then \( I \leq I + Q \) and \( r(I + Q) = 1 \). The assumption about order continuity of the operator \( T \) is also essential. It is sufficient to take in the previous example the operator \( Q \) such that \( Q \) is a band irreducible compact operator rank-one with \( r(Q) = 0 \) (see [2]).

Now let us pay attention to the case of integral operators. Next example shows that, even with the assumption of the integrality of the operator \( T \), theorem 5 doesn’t hold without the assumption concerning a pole at the point \( \lambda_0 = r(T) \). Recall that a bounded operator \( Q \) on a Hilbert space \( H \) is Hermitian, if \( Q^* = Q \). For a Hermitian operator \( Q \) the equality ([1], p. 465) \( r(Q) = \|Q\| \) holds.

**Example 6.** Consider the space \( \ell_2 \). Let \( S \) be an operator on \( \ell_2 \) defined by the diagonal matrix with elements \( s_{ii} = 1 - \frac{1}{2^i} \) on the diagonal. Clearly, \( r(S) = 1 \). If we show that there exists a sequence of real numbers \( a_n > 0 \) such that the operator \( T \), defined by the infinity matrix

\[
 \begin{pmatrix}
 \frac{1}{2} & a_1 & a_2 & a_3 & \ldots \\
 a_1 & \frac{3}{4} & 0 & 0 & \ldots \\
 a_2 & 0 & \frac{7}{8} & 0 & \ldots \\
 a_3 & 0 & 0 & \frac{15}{16} & \ldots \\
 & \ldots & \ldots & \ldots & \ldots
\end{pmatrix},
\]

acts on \( \ell_2 \) and \( r(T) = 1 \), then, as a result, \( 0 \leq S < T \), the operator \( T \) is ideal irreducible integral and \( r(S) = r(T) = 1 \).

By induction we find a decreasing sequence of numbers \( a_n > 0 \) such that for all \( n \) spectral radii of matrices

\[
 \begin{pmatrix}
 \frac{1}{2} & a_1 & \ldots & a_n \\
 a_1 & \frac{3}{4} & \ldots & 0 \\
 & \ldots & \ldots & \ldots \\
 a_n & 0 & \ldots & 1 - \frac{1}{2^{n+1}}
\end{pmatrix}
\]
are strictly less of unit. Then \( \sum_{i=1}^{\infty} a_i^2 \leq \frac{3}{4} \). Show that by such choice of \( a_n \) \( r(T) = 1 \). Let \( T_n \) be operators defined by matrices

\[
\begin{pmatrix}
\frac{1}{2} & a_1 & \ldots & a_n & 0 & 0 & \ldots \\
a_1 & \frac{1}{2} & \ldots & 0 & 0 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
a_n & 0 & \ldots & 1 - \frac{1}{2^{n+1}} & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 1 - \frac{1}{2^{n+2}} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}
\]

Then \( r(T_n) = 1 \). In fact, the inequality \( r(T_n) \geq 1 \) is obvious. For the converse

\[
\|T_n x\|^2 = \|T_n P_{n+1} x\|^2 + \|T_n (I - P_{n+1}) x\|^2 \leq \|P_{n+1} x\|^2 + \|(I - P_{n+1}) x\|^2 = \|x\|^2,
\]

where \( x = (x_1, x_2, \ldots) \geq 0 \), \( P_n \) are projections on first \( n \) coordinates, hence it follows relations \( r(T_n) = \|T_n\| \leq 1 \). Thus \( r(T_n) = 1 \).

Further, the sequence \( T_n \) converges to the operator \( T \) in \( L(\ell_2, \ell_2) \). Actually

\[
\|(T - T_n)x\|^2 = \|\sum_{i=n+1}^{\infty} a_i x_{i+1}, 0, \ldots, 0, a_{n+1} x_{n+2}, \ldots\|^2 \leq \left( a_{n+1}^2 + \sum_{i=n+1}^{\infty} a_i^2 \right) \|x\|^2,
\]

hence \( \|T - T_n\|^2 \leq a_{n+1}^2 + \sum_{i=n+1}^{\infty} a_i^2 \to 0 \) by \( n \to \infty \).

Thus

\[
r(T) = \|T\| = \lim_{n \to \infty} \|T_n\| = \lim_{n \to \infty} r(T_n) = 1.
\]

We can choose the sequence \( a_n \) in example 6 like that: for an ideal irreducible integral operator \( T \) with \( r(T) = 1 \) there exists an element \( x > 0 \) such that \( Tx < x \). Note that in example 6 an element \( x > 0 \) such that \( Tx > x \) doesn’t exist.

Next theorem suggested the idea of the construction of example 6.

**Theorem 7.** Let \( T \) and \( K \) be Hermitian operators on \( L_2 \) such that \( 0 \leq K < T \), the operator \( K \) is compact and the point \( \lambda_0 = r(T) \) doesn’t belong to the point spectrum \( \sigma_p(T) \) of the operator \( T \). Then \( r(T - K) = r(T) \).

**Proof.** From the equality \( \sigma_r(T) = \sigma_p(T) \), where \( \sigma_r(T) \) is the residual spectrum of the operator \( T \), it follows that the point \( \lambda_0 = r(T) \) belongs to the continuous spectrum of the operator \( T \). By Weyl theorem ([3], p. 320) we have \( r(T) \in \sigma(T - K) \), hence \( r(T) \leq r(T - K) \), therefore \( r(T) = r(T - K) \).

Thus for Hermitian operators \( T \geq 0 \) on \( L_2 \) the condition that \( r(T) \) is a pole of \( R(., T) \), namely \( r(T) \in \sigma_p(T) \), is necessary in order that the inequalities \( 0 \leq S < T \) imply the inequality \( r(S) < r(T) \). Note that, as a matter of fact, we can formulate theorem 7 for a wide class of operators on some Banach lattice \( E \), for which Weyl theorem about a perturbation of the spectrum is true.

Note also that conditions, when for an irreducible Hermitian operator \( T \) and a diagonal operator \( D \) on \( \ell_2 \) either \( r(T + D) = r(T) \) or \( r(T + D) > r(T) \) hold, are studied in [13].

Same examples as example 6, exist and in more classical Banach lattices, namely in \( \ell_1 \) and \( \ell_\infty \). To the proof this assertion, we need a next lemma, which generalizes lemma V.6.4 from [15]. The proof of it is analogous and will be omitted.
Lemma 8. Let $E$ be a Banach lattice, $S$ and $T$ be operators on $E$ such that $0 \leq S \leq T$ and $r(S) = r(T)$ hold. Then, there exists an element $z > 0$ such that the ideal $I_z$ generated by $z$ is invariant under $S$ and $T$ and $r(S_z) = r(T_z) = r(T)$, where $S_z$ and $T_z$ are restrictions of operators $S$ and $T$ on $M$-space $I_z$, respectively.

Example 9. Let $S$ and $T$ be operators from example 6. Use lemma 8 to find the element $z > 0$ such that the ideal $I_z$ is invariant under $S$ and $T$ and moreover $r(S_z) = r(T_z) = 1$. Define an operator $T_\infty$ on $\ell_\infty$ by the rule $T_\infty x = \frac{1}{z}T(xz)$. For an arbitrary element $x \in \ell_\infty$
\[
(T_\infty x)_i = \frac{1}{z_i}(T(xz))_i = \frac{1}{z_i} \sum_{j=1}^{\infty} a_{ij}x_j z_j = \sum_{j=1}^{\infty} \frac{z_j}{z_i}a_{ij}x_j
\]
holds, where $K = \{(a_{ij})\}$ is the kernel of the operator $T$. Therefore $\{(\frac{z_j}{z_i}a_{ij})\}$ is the kernel of the operator $T_\infty$ and thus, the operator $T_\infty$ is band irreducible integral. From equalities $T_\infty e = \frac{1}{z}T^n z$ and $\|T^n z\|_\infty = \|x\|_z$, where $e = (1, 1, \ldots)$, that are holding for all natural $n$ and all elements $x \in I_z$, we obtain
\[
\|T_\infty\|_\infty^{\frac{1}{n}} = \|\frac{1}{z}T^n z\|_\infty^{\frac{1}{n}} = \|T^n z\|_z^{\frac{1}{n}} = \|T_z\|_z^{\frac{1}{n}} \rightarrow 1,
\]
whence $r(T_\infty) = 1$ (as a matter of fact, we can show that $\sigma(T_\infty) = \sigma(T_z)$). Analogously we establish that $r(S_\infty) = 1$, where $S_\infty$ is an operator on $\ell_\infty$ defined by rule $S_\infty x = \frac{1}{z}S_z(xz)$.

Thus on the space $\ell_\infty$ there exist a band irreducible integral operator $T$ and a positive operator $S$ such that $S < T$ and $r(S) = r(T)$. Considering the restrictions of operators $S^*$ and $T^*$ on $\ell_1$, we obtain the analogous example in space $\ell_1$.

Note that by a more “precise” choice of an element $z$, we can get even the ideal irreducibility of an operator $T$.

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