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Decomposing approach to multi-commodity network flow programming problems
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Abstract: We propose the theory of decomposition, methods, technologies, applications and pseudocodes of algorithms for constructing the optimal solution of the inhomogeneous network flow problems.

Keywords: Network optimization, multi-commodity, decomposition, multigraph, graph partitioning technology, sparse linear system, root tree, basis of the solution space, pseudocode.

1. INTRODUCTION
This work is devoted to the research of intelligent multi-commodity transport systems and their applications. The investigated problem concerns the class of network multi-commodity optimization problems, which has the additional parameters and constraints.

In this work we consider the application of the graph theory and network flow optimization theory for construction the optimal solution of the inhomogeneous network flow programming problems. Technologies and algorithms for the solution of problems with high dimensional data are developed. The algorithms for finding the optimal solution of the inhomogeneous network flow programming problems are based on constructive theory of decomposition, the theoretic-graph specificities of the structure of the support for the multigraph and on the properties of the basis of the solution space for the homogeneous sparse systems of special types. Decomposing approach is allowed us to apply parallelization of the computational process for finding the optimal solution of the inhomogeneous network flow programming problems.

2. MATHEMATICAL MODEL
Let $G=(V,A)$ be a finite oriented connected multigraph without loops, where $V$ is a set of nodes and $A$ is a set of arcs defined on $V \times V$ $(|V|<\infty, |A|<\infty)$. Let $K(|K|<\infty)$ be a set of different products (types of flow) transported through the network $G$. For definiteness, we assume the set $K=[1,\ldots,|K|]$. Let us denote the connected network corresponding to a certain type $k$ of flow with $S^k=(I^k, U^k)$, where $I^k$ is the set of nodes and $U^k$ is the set of arcs which is available on the flow of type $k$, $k \in K$. Also, we define for each node $i \in V$ the set of types of flows $K(i) = \{ k \in K : i \in I^k \}$ and for each arc $(i,j) \in A$ the set $K(i,j) = \{ k \in K : (i,j)^k \in U^k \}$. In other words, $K(i)$ is the set of types of flows transported through the node $i \in V$ and $K(i,j)$ is the set of types of flows transported through the multiarc $(i,j) \in A$ respectively. Finally, the initial network $G=(V,A)$ may be considered as a union of $|K|$ networks $S^k=(I^k, U^k)$, $k \in K$. We call $G=(V,A)$ a multigraph or multigraph or multinetwork or just a network, if it is clear that we deal with multigraph $G$. Each multiarc $(i,j) \in A$ of multigraph $G$ consists of $|K(i,j)|$ arcs $(i,j)^k$, $k \in K(i,j)$.

Let us introduce a subset $U_0$ of the set $A$, and let $K_0(i,j) \subseteq K(i,j)$, $(i,j) \in U_0$ be an arbitrary subset of $K(i,j)$ such that $|K_0(i,j)|>1$.

We apply the decomposing approach to constructing the optimal solution of minimum cost multi-commodity problem with embedded network structure of constraints.

We consider the following mathematical model:

$$\sum_{(i,j) \in E} \sum_{k \in K} c_{ij}^k x_{ij}^k \rightarrow \min, \quad (1)$$

$$\sum_{j \in I^k} x_{ij}^k = d_i^k, \quad i \in I^k, \quad k \in K, \quad (2)$$

$$\sum_{(i,j) \in E} \sum_{k \in K} \lambda_{ij}^k x_{ij}^k = \alpha_p, \quad p = \bar{1},q, \quad (3)$$

$$\sum_{k \in K(i,j)} x_{ij}^k \leq d_{ij}^k, \quad x_{ij}^k \geq 0, \quad k \in K_0(i,j), (i,j) \in U_0, \quad (4)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, \quad k \in K(i,j), (i,j) \in A, \quad (5)$$

$$x_{ij}^k \geq 0, \quad k \in K(i,j), K_0(i,j), (i,j) \in A \setminus U_0, \quad (6)$$

where $c_{ij}^k, d_i^k, \lambda_{ij}^k, d_{ij}^k, \alpha_p$ - parameters, $I^k(\bar{U}^k) = \{ j \in I^k : (i,j)^k \in \bar{U}^k \}$, $I^k(U^k) = \{ j \in I^k : (i,j)^k \in U^k \}$, $x = (x_{ij}^k, (i,j)^k) \in U^k, k \in K$ - vector of unknowns.

3. GENERAL SOLUTION OF SPARSE UNDERDETERMINED SYSTEMS
We introduce the support of multigraph $G=(V,A)$ for the problem (1) - (6) as a set $U_S$ [1]:

$$U_S = U_0 \cup U^k, \quad k \in K, \quad U^k \subset U^k, \quad k \in K; \quad (7)$$

$$U^\ast \subseteq U_0, \quad U_0^\ast = \{ (i,j) \in U_0 : |K^\ast(i,j)|>1 \}, \quad (8)$$

$$K_S(i,j) = \{ k \in K(i,j) : (i,j)^k \in U^k \}, \quad (i,j) \in A, \quad (9)$$

$$K^\ast_S(i,j) = K_S(i,j) \cap K_0(i,j), \quad (i,j) \in U_0, \quad (10)$$

such the following system

$$\sum_{(i,j) \in E} \sum_{k \in K} c_{ij}^k x_{ij}^k \rightarrow \min, \quad (11)$$

$$\sum_{j \in I^k} x_{ij}^k = d_i^k, \quad i \in I^k, \quad k \in K, \quad (12)$$

$$\sum_{(i,j) \in E} \sum_{k \in K} \lambda_{ij}^k x_{ij}^k = \alpha_p, \quad p = \bar{1},q, \quad (13)$$

$$\sum_{k \in K(i,j)} x_{ij}^k \leq d_{ij}^k, \quad x_{ij}^k \geq 0, \quad k \in K_0(i,j), (i,j) \in U_0, \quad (14)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, \quad k \in K(i,j), (i,j) \in A, \quad (15)$$

$$x_{ij}^k \geq 0, \quad k \in K(i,j), K_0(i,j), (i,j) \in A \setminus U_0, \quad (16)$$

where $c_{ij}^k, d_i^k, \lambda_{ij}^k, d_{ij}^k, \alpha_p$ - parameters, $I^k(\bar{U}^k) = \{ j \in I^k : (i,j)^k \in \bar{U}^k \}$, $I^k(U^k) = \{ j \in I^k : (i,j)^k \in U^k \}$, $x = (x_{ij}^k, (i,j)^k) \in U^k, k \in K$ - vector of unknowns.

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\[ \sum_{j \in L_k} x_{ij}^k - \sum_{j \in L_k} x_{jil}^k = 0, \quad i \in I^k, \quad k \in K, \quad (7) \]

\[ \sum_{k \in K} \sum_{i \in J} x_{ij}^k = 0, \quad p = 1, q, \quad (8) \]

\[ \sum_{k \in K} x_{ij}^k = 0, \quad (i, j) \in \tilde{U}^* \quad (9) \]

has only a trivial solution \( x_{ij}^k = 0 \), \((i, j)^k \in \tilde{U}^*\) for \( \tilde{U}^* = U^*_k, \quad k \in K \). \( \tilde{U}^* = U^* \setminus \{i, j\}_k \), but has a non-trivial solution if the following condition are true:

1) \( \tilde{U}^* = U^*_k, \quad k \in K \); \( \tilde{U}^* = U^* \setminus \{i, j\}_k \);

\((i, j)_k \in U^*_k, \quad K_2^k(i, j) = K_2^k(i, j), \quad (i, j) \in U^* \);

2) \( \tilde{U}^* = U^*_k, \quad k \in K \); \( \tilde{U}^* = \bigcup_{K} U^*_k \cup \{i, j\}_k \), \( \tilde{U}^* = U^*, \quad (i, j)_k \in U^* \quad (10) \)

where

\[ K_2^k(i, j) = \begin{cases} K_2^k(i, j), & \text{if} \quad (i, j)^k \notin \{i, j\}_k, \\ K_2^k(i, j) \cup (i, j)_k \cup (i, j)_k, & \text{if} \quad (i, j)^k = \{i, j\}_k. \end{cases} \]

The set \( U^*_k \) contains \( K(k) \) arcs \((i, j)^k \), the deletion of which gives the set \( U^*_k \) such that the network \( (I, U^*_k, U^*_k) \) doesn’t contain cycles. Each network \( (I, U^*_k, U^*_k) \) contains a single cycle, \((i, j)^k \notin \tilde{U}^*_k, \quad k \in K \). We denote this cycle as \( L^*_k \). The set of arcs

\( L^*_k = \{U^*_k, k \in K \} \)

is a forest of spanning trees. We define a set of cycle arcs \( U^*_k = \bigcup_{k \in K} U^*_k \), \( k \in K \) by selecting |\( U^*_k = q^+ | U^*_0 \)| arbitrary arcs from the sets \( U^*_k \setminus U^*_k \), \( k \in K \).

We represent the general solution of system (2) for a fixed \( k \in K \) as a sum of general solution of homogeneous system corresponding to the system (2) and particular solution of the inhomogeneous system (2). The general solution of system (2) for a fixed \( k \in K \) has the following form:

\[ x_{ij}^k = \sum_{(i, j)^k \in U^*_k} x_{ij}^{\tau} \text{sign}(i, j)^{\tau} + \]

\[ \text{sign}(i, j)^{\tau} \in \{1, 0, -1 \}, \quad \sum_{(i, j)^k \in U^*_k} x_{ij}^{\tau} = 0 \quad (10) \]

\((i, j)^k \in U^*_k, \quad (i, j)^k \in U^*_k, \quad x_{ij}^{\tau} \in (1), \quad \text{where} \quad \tilde{x} = (x_{ij}^{\tau}, (i, j)^k \in U^*_k) \) is any particular solution of the inhomogeneous system (2); \( \text{sign}(i, j)^{\tau} \) is the sign of an arc \((i, j)^k \) within the cycle \( L^*_k \) [2].

The efficient algorithms for constructing a general solution of homogeneous system corresponding to the system (2) in the form of a linear combination of characteristic vectors

\[ \delta^k(\tau, \rho) = (\delta^k_0(\tau, \rho), (i, j)^k \in U^*_k) \]

generated by an arc \((\tau, \rho)^k \in U^*_k, \quad k \in K \), and for constructing a particular solution have been developed [3,4].

4. TECHNOLOGIES AND ALGORITHMS FOR CONSTRUCTING THE BASIS OF THE SOLUTION SPACE

We present data structures for representing the root tree. Each root tree \((I, U^*_k, U^*_k) \) with root node \( \text{root} \) is constructed in accordance with a bijective mapping \[2,5\] for the corresponding spanning tree \((I, U^*_k, U^*_k) \).

So we use the following data structures [2]:

- \( \text{parent}^i = \{\text{parent}^i[i], i = 1, 1^k \} \) - the list of parents of the nodes in the root tree, \( \text{parent}[^\text{root}] = 0 \);

- \( \text{arc}^i = \{\text{arc}^i[i], i = 1, 1^k \} \) - the list of the arcs' directions of the root tree in accordance with the bijective mapping. This list determines the direction of the arcs \( \{\text{parent}^i[i], i = 1, 1^k \} \) of the root tree in accordance with the direction of the arcs of the spanning tree: \( \text{arc}^i[i] = 1, \) if the arc \( \{\text{parent}^i[i], i = 1, 1^k \} \); \( \text{arc}^i[i] = -1, \) if the arc \( \{\text{parent}^i[i], i = 1, 1^k \} \);

- \( \text{depth}^i = \{\text{depth}^i[i], i = 1, 1^k \} \) - the list of depths of root trees' nodes, \( \text{depth}[^\text{root}] = 0 \);

- \( \text{thread}^i = \{\text{thread}^i[i], i = 1, 1^k \} \) - the list of indices of communication;

- \( \text{invertThread}^i = \{\text{invertThread}^i[i], i = 1, 1^k \} \) - the list of the inverted order of dynamic bypass of root tree without root node.

In Fig.1 we present the pseudocode of the algorithm for computing the nonzero components of the characteristic vector \( \delta^k(\tau, \rho) = (\delta^k_0(\tau, \rho), (i, j)^k \in U^*_k) \), generated by an arc \((\tau, \rho)^k \in U^*_k, \quad k \in K \) [4]. The number of operations to compute each characteristic vector is proportional to \( |I^k| \) in the worst case.

**List of the reference symbols (Fig. 1):**

- \( k \) - type of flow;
- \( U^k \) - the set of arcs of the flow type \( k \);
- \( U^*_k \) - the arcs' set of the spanning tree of the flow type \( k \);
- \( \text{arc}^i \) - the arc of set \( U^*_k \); \( U^*_k \);
- \( \delta^k(\tau, \rho) \) - characteristic vector, generated by an arc \((\tau, \rho)^k \), with components \( \delta^k_0(\tau, \rho) \);
diff - difference of the depths of nodes $\tau$ and $\rho$;

sign - variable that corrects the sign of the arc direction in the cycle.

Bxo{\text{d}}: $k, (\tau, \rho), \text{parent}_{k}, \text{dir}_{k}, \text{depth}_{k}$

Bxo{\text{d}}: $\delta_{k}(\tau, \rho)$

1. if $\text{depth}_{k}[\tau] \geq \text{depth}_{k}[\rho]$ then
2. $\text{diff} \leftarrow \text{depth}_{k}[\tau] - \text{depth}_{k}[\rho]$
3. $t_{1} \leftarrow \tau, t_{2} \leftarrow \rho, \text{sign} \leftarrow 1$
4. else
5. $\text{diff} \leftarrow \text{depth}_{k}[\rho] - \text{depth}_{k}[\tau]$
6. $t_{1} \leftarrow \rho, t_{2} \leftarrow \tau, \text{sign} \leftarrow -1$
7. end if
8. while $\text{diff} \neq 0$
9. if $\text{dir}_{k}[t_{1}] = 1$ then
10. $\delta_{k_{\text{parent}_{k}[t_{1}]}, t_{1}} \leftarrow \text{sign}$
11. else
12. $\delta_{k_{\text{parent}_{k}[t_{1}]}, t_{1}} \leftarrow -\text{sign}$
13. end if
14. $t_{1} \leftarrow \text{parent}_{k}[t_{1}]
15. \text{diff} \leftarrow \text{diff} - 1$
16. end while
17. if $t_{1} \neq t_{2}$ do
18. if $\text{dir}_{k}[t_{1}] = 1$ then
19. $\delta_{k_{\text{parent}_{k}[t_{1}], t_{1}}} \leftarrow 1$
20. else
21. $\delta_{k_{\text{parent}_{k}[t_{1}], t_{1}}} \leftarrow -1$
22. end if
23. $t_{1} \leftarrow \text{parent}_{k}[t_{1}]
24. \text{diff} \leftarrow \text{diff} - 1$
25. end while
26. end if
27. return $\delta_{k}(\tau, \rho)$

Fig.1 – Pseudocode of the algorithm for computing the nonzero components of the vector of the solution space’s basis.

5. PARTICULAR SOLUTION

The pseudocode of the algorithm for computing the particular solution $\tilde{x}^{k} = (\tilde{x}^{k}_{ij}, (i, j) \in U^{k})$ of system (2) is represented on Fig.2.

List of the reference symbols (Fig. 2):

$k$ - type of flow;

$U^{k}$ - the set of arcs of the flow type $k$;

$U_{r}^{k}$ - the arcs’ set of the spanning tree of the flow type $k$;

$q^{i}$ - intensity of node $i \in I^{k}$ for the flow type $k$;

$\varphi[i]$ - the marker of node $i \in I^{k}$;

$\tilde{x}^{k}$ - the vector of numerical values for particular solution with components $\tilde{x}^{k}_{ij}$;

inverseThread - the list of the inverted order of dynastic bypass of root tree without route node.

Bxo{\text{i}}: $k, \text{parent}_{k}, \text{dir}_{k}, \text{inverseThread}_{k}$

Bxo{\text{i}}: $\tilde{x}^{k}$

1. for $(i, j) \in U^{k}$ do
2. $\tilde{x}^{k}_{ij} \leftarrow 0$
3. end for
4. for each element $j$ in $\text{inverseThread}_{k}$ do
5. $p \leftarrow \text{parent}_{k}[j]$
6. $x_{\text{tmp}} \leftarrow 0$
7. if $\text{dir}_{k}[j] = 1$ then
8. $\tilde{x}^{k}_{p, j} \leftarrow \tilde{x}^{k}_{p, j} - a_{j}^{k}$
9. $x_{\text{tmp}} \leftarrow x_{\text{tmp}} + \tilde{x}^{k}_{p, j}$
10. end if
11. if $\text{dir}_{k}[j] = -1$ then
12. $\tilde{x}^{k}_{p, j} \leftarrow \tilde{x}^{k}_{p, j} + a_{j}^{k}$
13. $x_{\text{tmp}} \leftarrow -x_{\text{tmp}}$
14. end if
15. if $\text{dir}_{k}[p] = 1$ then
16. $\tilde{x}^{k}_{p, \text{parent}_{k}[p], p} \leftarrow \tilde{x}^{k}_{p, \text{parent}_{k}[p], p} + x_{\text{tmp}}$
17. end if
18. if $\text{dir}_{k}[p] = -1$ then
19. $\tilde{x}^{k}_{p, \text{parent}_{k}[p], p} \leftarrow \tilde{x}^{k}_{p, \text{parent}_{k}[p], p} - x_{\text{tmp}}$
20. end if
21. end for
22. end for
23. return $\tilde{x}^{k}$

Fig.2 – Pseudocode of the algorithm for computing the particular solution.

6. OBJECTIVE FUNCTION INCREMENT

Let $\{x, U_{s}\}$ is a pair consisting of an arbitrary multiflow $x$ and an arbitrary support $U_{s}$ of the problem $(1)-(6)$, $x = (x^{k}_{ij}, (i, j) \in U_{s}, k \in K(i, j))$. We consider some other multiflow $\bar{x}^{k} = x^{k} + \Delta x^{k}$, $\bar{x} = (\bar{x}^{k}_{ij}, (i, j) \in A, k \in K(i, j))$, where $\Delta x = (\Delta x^{k}_{ij}, (i, j) \in A, k \in K(i, j))$ – increment of multiflow $x$.

The objective function increment formula of the problem (1) has a form:

$$\Delta \phi(x) = \sum_{(i, j) \in U^{k}} \gamma_{ij} \Delta x^{k}_{ij} + \sum_{k \in \Phi} \sum_{(i, j) \in U^{k}} \Delta x^{k}_{ij} \Delta y^{k}_{ij},$$

$$\Delta y^{k}_{ij} = \delta^{k}_{ij} - \sum_{p \in U^{k}} \gamma^{k}_{ij} \Delta y^{k}_{ij},$$

where

$$\gamma^{k}_{ij} = \delta^{k}_{ij}, (i, j) \in U^{k}, k \in K,$$

$$\delta^{k}_{ij} = \sum_{(i, j) \in U^{k}} c^{k}_{ij} \text{sign}(i, j)^{k},$$

$$(i, j) \in U^{k}, \text{sign}(i, j)^{k} \in \{-1, 1\},$$

$$(i, j, k) \in U^{k}, (i, j) \in A, k \in K,$$

The values $\Delta y^{k}_{ij}$ we call the estimates.
7. CONDITIONS OF OPTIMALITY
We introduce the concept of nondegeneracy of the support multiflow \( \{x, U_S\} \). The support multiflow \( \{x, U_S\} \) is called non-degenerate if the following conditions are satisfied:

\[
0 < x^0_j < d^0_j, \quad k \in K^0(i,j), \quad (i,j) \in A, \\
x^0_j > 0, \quad k \in K^0(i,j) \setminus K^0(i,j), \quad (i,j) \in A \setminus U_0, \\
x^0_j > 0, \quad k \in K^0(i,j), \quad (i,j) \in U_0, \\
0 < \sum_{k \in K(i,j)} x^0_j < d^0_j, \quad (i,j) \in U_0 \setminus U^*.
\]

For optimality of the support multiflow \( \{x, U_S\} \), it is necessary, and in the case of non-degeneracy (13), it is sufficient to satisfy the conditions:

\[
x^0_j = 0, \quad \text{if } \Delta^0_j > 0, \\
x^0_j = d^0_j, \quad \text{if } \Delta^0_j < 0, \\
x^0_j \in [0, d^0_j], \quad \text{if } \Delta^0_j = 0, \quad k \in K^0(i,j), \quad (i,j) \in A; \\
x^0_j = 0, \quad \text{if } \Delta^0_j > 0, \quad k \in K^0(i,j), \quad (i,j) \in U_0; \\
x^0_j \geq 0, \quad \text{if } \Delta^0_j = 0, \quad k \in K^0(i,j), \quad (i,j) \in U_0; \\
x^0_j = 0, \quad \text{if } \Delta^0_j > 0, \quad k \in K^0(i,j) \setminus K^0(i,j), \quad (i,j) \in A \setminus U_0; \\
x^0_j \geq 0, \quad \text{if } \Delta^0_j = 0, \quad k \in K^0(i,j) \setminus K^0(i,j), \quad (i,j) \in A \setminus U_0; \\
\sum_{k \in K(K(i,j))} x^0_j = 0, \quad \gamma^0_j > 0, \\
\sum_{k \in K(K(i,j))} x^0_j = 0, \quad \gamma^0_j < 0, \\
\sum_{k \in K(K(i,j))} x^0_j \in [0, d^0_j], \quad \gamma^0_j = 0, \quad (i,j) \in U^*.
\]

8. DECOMPOSITION OF SUITABLE DIRECTION
We present the suitable direction \( y = (y^0_j, (i,j) \in A) \), \( y^0_j = (y^0_j, k \in K(i,j)) \) [1, 5] of the multiflow \( x \) change according to the simplex normalization. The adaptive normalizations is applied in [6]. The following cases are possible:

1) \( p_0 = 1, \quad k_0 = K^0(i_j, j_0) \); 
2) \( p_0 = 2, \quad (i_0, j_0) \in U^* \).

Consider case 1). Components \( y^0_j, k \in K^0(i_j, j), \quad (i,j) \in A \); \( x^0_{k_0} \) of suitable direction \( y \) satisfy to the system:

\[
\sum_{j \in I^k} y^0_j - \sum_{j \in I^k} y^0_j = 0, \quad i \in I^k, \quad k \in K \setminus k_0. 
\]

\[
\sum_{j \in I^k \cup I^0} y^0_j - \sum_{j \in I^k \cup I^0} y^0_j = 0, \quad i \in I^k. 
\]

\[
\sum_{k \in K} \sum_{(i,j) \in E(k)} y^0_j + \sum_{(i,j) \in E(k)} y^0_j = 0, \quad i \in I^k, \quad k \in K. 
\]

\[
p = \frac{1}{q},
\sum_{k \in K(i,j)} y^0_j = 0, \quad (i,j) \in U^*, \quad y^0_j = -\text{sgn}(\Delta^0_j),
\]

where

\[
\hat{K}_S(i,j) = \left\{ \begin{array}{ll}
K_S(i,j), & \text{if } (i,j) = (i_0, j_0), \\
K_S(i-j, j) \cup k_0, & \text{if } (i,j) = (i_0, j_0), \\
K_S(i-j, j) \cup (k_0 \setminus K_S(i-j, j)), & \text{if } (i,j) = (i_j, j_0),
\end{array} \right.
\]

or

\[
\hat{K}_S(i,j) = \hat{K}_S(i,j) \cap K_S(i,j), \\
\hat{K}_S(i,j) = \hat{K}_S(i,j) \cap K_S(i,j).
\]

We present an efficient algorithm for solving the system (14)–(17), based on the decomposing approach. As a result of the decomposition with respect to variables corresponding to the arcs of the sets \( U_C \) and \( U_P \), independent systems are obtained. Consider the fundamental cycle \( L^0_{k_0} \), generated by the arc \( (i_0, j_0) \), relative to the arcs of the spanning tree \( U^0_{k_0} \). Let \( L^0_{k_0} \) be the sets of forward arcs and backward arcs of the cycle \( L^0_{k_0} \) respectively. For the cyclic arcs of the set \( U_C \) of the support \( U_S \) the following relations are true:

\[
\sum_{k \in K \setminus i, j} R^0_{i, j} y^0_j = R^0_{i, j} y^0_{k_0} \text{sgn}(\Delta^0_j_{k_0}), \quad p = \frac{1}{q};
\]

\[
\sum_{k \in K \setminus i, j} \delta^0_{i, j} y^0_j = \delta^0_{i, j} y^0_{k_0} \text{sgn}(\Delta^0_j_{k_0}), \quad (i,j) \in U^*.
\]

We denote \( D = D(U_S) \) the matrix of determinant, which corresponds to the support \( U_S = \{U^0_k, k \in K, U^* \} \) [1, 5]. So we calculate the components \( y^0_j, (i,j) \in U_C, \quad k \in K \) of the suitable direction \( y \), which correspond to the cyclic arcs. The components of the vector \( y^0_j \), which corresponding to the arcs of the spanning tree \( U^0 \), we calculate from the balance conditions (14)–(15) for each \( k \in K \) using the properties of the root tree corresponding to the spanning tree \( U^0 \). The remaining components of the suitable direction \( y \) are calculated as follows: \( y^0_j = 0, \quad k \in (K(i,j) \setminus K_S(i,j), (i,j)) \in A \).

Consider case 2). The support components \( y^0_j, k \in K^0(i_j, j), \quad (i,j) \in A \) of the suitable direction \( y \) of multiflow \( x \) change satisfy to the system:

\[
\sum_{j \in I^k} y^0_j - \sum_{j \in I^k} y^0_j = 0, \quad i \in I^k, \quad k \in K.
\]
Let us construct an efficient algorithm for solving the system (23). The potentials \( r = (r_p, p = 1, ..., p, \quad (i,j) \in U^* \) are calculated from the following linear system:

\[
D'^r = \omega,
\]

where

\[
\omega = (\omega_i, \quad i, j),
\]

\[
\omega_i = \sum_{\bar{i}, (i,j) \in \bar{U}^*} c_{\bar{i}, i} - \sum_{\bar{i}, (i,j) \in \bar{U}^*} c_{\bar{i}, j},
\]

\( \bar{r} = (\bar{r}_p, \quad p = 1, ..., p, \quad (i,j) \in \bar{U}^* \) is a cycle \( L'_p \) number, \( i \) is the number of fundamental cycles generated by arcs \( (\tau, \rho)_p \) from the set \( U_c \).

The system (24) has a unique solution because \( \overline{R}(U'_c) = \det D' \neq 0 \) [5]. We set \( u^+_i = 0 \) for selected \( i \in I^* \) for each \( k \in K \). The remaining components of the vectors \( u^* = (u^+_i, i \in I^*) \), \( k \in K \) we compute from the following system:

\[
u^+_i = \omega_i = -\sum_{p=1}^{\lambda_{\bar{r}}(L'_{p})} \delta_{\bar{r}}(L'_{p}) \quad \gamma_{k(i)} = 0, \quad (i,j) \in U^* \setminus \{i_0, j_0\},
\]

Since the matrix of determinants \( D \) is nondegenerate, then we calculate the components \( y_{k(i)}^+, \quad (i,j) \in U^*_k, \quad k \in K \) of the suitable direction \( y \) of multilow \( \chi \) change from the system (22). The components of the vector \( y(U^*_k) \), which correspond to the arcs of the spanning tree \( U^*_k \), we calculate from the balance conditions (18) for each \( k \in K \) using the properties of the root tree corresponding to the spanning tree \( U^*_k \). The remaining components of the suitable direction \( y \) are calculated as follows: \( y_{k(i)}^+ = 0, \quad k \in K(i,j) \), \( \hat{K}_k(i,j), \quad (i,j) \in A \).

9. DECOMPOSITION OF POTENTIALS SYSTEM

The potentials \( u^* = (u^+_i, i \in I^*) \), \( k \in K \) is a satisfying the following system of linear algebraic equations:

\[
u^+_i - u^+_i = \sum_{p=1}^{\lambda_{\bar{r}}(L'_{p})} \lambda_{\bar{r}}(L'_{p}) r_p + \gamma_{k(i)}^+, \quad (i,j) \in A \setminus U^*,
\]

\[
u^+_i - u^+_i = \sum_{p=1}^{\lambda_{\bar{r}}(L'_{p})} \gamma_{k(i)} = 0, \quad (i,j) \in A \setminus U^*,
\]

\[
u^+_i - u^+_i = -\sum_{p=1}^{\lambda_{\bar{r}}(L'_{p})} \lambda_{\bar{r}}(L'_{p}) r_p + \gamma_{k(i)}^+, \quad (i,j) \in U^*,
\]

\[
u^+_i - u^+_i = -\sum_{p=1}^{\lambda_{\bar{r}}(L'_{p})} \gamma_{k(i)} = 0, \quad (i,j) \in U^*.
\]

The number of operation of finding the components of vector \( u^* = (u^+_i, i \in I^*) \) is \( O(\mid I^* \mid) \) in the worst case, \( k \in K \).

10. EXAMPLE OF MULTI-COMMODITY NETWORK FLOW PROGRAMMING PROBLEM

In Fig. 3 we present the multinetowrk \( G = (V, A), \quad |V| = 4, \quad |A| = 6, \quad |K| = 2 \). The arcs with the first type of flow are shown by a solid line. The arcs with the second type of flow are shown by a dashed line.

We consider the following mathematical model:

\[
\begin{align*}
6x_{13} + 5x_{14} + 4x_{15} + 6x_{21} + 7x_{23} + 9x_{24} + 7x_{25} + 7x_{31} + 9x_{32} + 7x_{35} + 7x_{41} + 9x_{42} + 7x_{45} & \rightarrow \min, \\
x_{13} + x_{14} - x_{15} = 4, & x_{13} + x_{14} - x_{15} = 4, \\
x_{23} + x_{24} + x_{25} = 19, & x_{23} + x_{24} + x_{25} = 19, \\
x_{31} - x_{32} - x_{35} = 8, & x_{31} - x_{32} - x_{35} = 8, \\
x_{41} - x_{42} - x_{45} = -15, & -x_{41} - x_{42} - x_{45} = -15, \\
x_{13} + x_{14} + x_{15} + x_{21} + 4x_{23} + 5x_{24} + 10x_{25} + 10x_{26} + 2x_{31} + 9x_{34} + 4x_{35} + 7x_{36} + 9x_{37} = 384,
\end{align*}
\]
The characteristics of the multinetwork (Fig.3) are presented in the Table 1.

<table>
<thead>
<tr>
<th>(i,j)</th>
<th>(1,3)</th>
<th>(1,4)</th>
<th>(2,1)</th>
<th>(2,3)</th>
<th>(2,4)</th>
<th>(3,4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>U_k</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>U_i</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>U_v</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>K(i,j)</td>
<td>(1,2)</td>
<td>(1,2)</td>
<td>(1,2)</td>
<td>(1,2)</td>
<td>(1,2)</td>
<td>(1,2)</td>
</tr>
<tr>
<td>K_v(i,j)</td>
<td>∅</td>
<td>(1,2)</td>
<td>∅</td>
<td>(1,2)</td>
<td>∅</td>
<td>(1,2)</td>
</tr>
</tbody>
</table>

The initial support $U_0$ of the problem (25) – (28) is presented in Fig.4.

Fig.5 – The optimal support of the problem (25) – (28).

The obtained theoretical and practical results are an important contribution to the solution of network optimization problems.

11. ACKNOWLEDGEMENT

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12. REFERENCES