Interpolating rational Bézier spline surfaces with local shape control

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Abstract: *The paper presents a technique for construction of C^n rational Bézier spline surfaces by interpolation of rectangular greed of points. The spline surfaces are constructed by blending rational Bézier curves using special polynomials. The constructed spline surfaces have local shape control that make them useful in on-line geometric application, fast surface sketching and modeling surfaces obtained from projections.*

***Keywords*:** Blending surfaces, rational Bézier surfaces, interpolation, splines surfaces, interpolating rational spline surfaces

1. MOTIVATION

In image processing and geometric modeling it is often necessary to reconstruct surfaces using grids of knot points. These problems are usually solved by construction of spline surfaces interpolating or approximating these grids [1, 2]. Classical approaches to construct spline surfaces on grids needs determination of conditions at knot points which ensure necessary continuity of spline surfaces. These conditions usually describe values of partial derivatives to which patches of spline surfaces must satisfy at knot points. But it is not easy to determine values of the partial derivatives which ensure proper shape and other geometric characteristics of spline surfaces. Usually to solve this problem some empirical assumptions or optimization criteria are used. To avoid the difficulties Bézier surfaces are still widely used to solve the problem because in this case continuity of spline surfaces is controlled by frame points. That is why the points are also called control points.

The paper presents a different approach to interpolation of rectangular grids by spline surfaces. The approach is based on smooth deformation of curves which form border of the patch. The deformation is fulfilled by means of polynomials which satisfy special boundary conditions. Using this approach the problem of determining conditions which ensure continuity of an interpolating surface at knot points is avoided. Besides shape of the surface can be controlled not only by knot points of the greed but also by weights which are assigned to the knot points.

The paper is structured as follows. Firstly polynomials for deformation of curves are briefly described. Detailed consideration of the polynomials can be found in the work [3, 4]. Next an approach to construction spline surfaces interpolating rectangular grids is presented.

1. PROBLEM STATEMENT

Consider a rectangular grid of points $\boldsymbol{p}_{ij}$, $i \in \{0, 1, \ldots, m\}$, $j \in \{0, 1, \ldots, n\}$, in an affine space. The problem is to construct a C^n continuous surface which interpolates points of this grid. Moreover it is required to provide local modification of the constructed surface shape, provided that the knot points are fixed.

Solution of the problem is presented as follows. Firstly construction of Bézier surface patches by deformation of its boundaries is considered. Then the approach is extended to rational Bézier surfaces patches. Such presentation facilitates understanding of the issue.

2. BLENDING POLYNOMIALS

The following polynomials will be used for modeling spline surfaces:

$$w_n(u) = \sum_{i=n}^{2n-1} b_{2n+1,\, i}(u)\,,\ u \in [0,1]\,, \tag{1}$$

where $b_{n,m}(u)$ denotes Bernstein polynomials:

$$b_{n,m}(u) = C_n^m (1-u)^{n-m} u^m\,.$$

It follows from this definition that the introduced polynomials $w_n(u)$ satisfy the following boundary conditions:

$$w_n(0) = 0\,,\ w_n(1) = 1\,, \tag{2}$$

$$w_n^{(m)}(0) = w_n^{(m)}(1) = 0 \tag{3}$$

for any $m \in \{1,2,\ldots,n-1\}$. It is proved [4] that the polynomial $w_n(u)$ is a minimum of the functional:

$$J_n(f) = \int_0^1 |\, f^{(n)}(u)\,|^2\ du$$

provided that the function $f(u)$ satisfies the boundary conditions (2) and (3).

3. MODELING SURFACE PATCHES BY BLENDING ITS BOUDARIES

Consider four parametric curves $\boldsymbol{p}_i(u)$, $u \in [0,1]$, $\boldsymbol{q}_i(v)$, $v \in [0,1]$, $i = 1,2$, in an affine space *A*, which have the following common boundary points:

$$\boldsymbol{p}_0(0) = \boldsymbol{q}_0(0) = \boldsymbol{r}_{0,0}\,,\ \boldsymbol{p}_0(1) = \boldsymbol{q}_1(0) = \boldsymbol{r}_{1,0}\,, \tag{4}$$

$$\boldsymbol{p}_1(0) = \boldsymbol{q}_0(1) = \boldsymbol{r}_{0,1}\,,\ \boldsymbol{p}_1(1) = \boldsymbol{q}_1(1) = \boldsymbol{r}_{1,1}\,. \tag{5}$$

The problem is to construct a rectangular patch $\boldsymbol{r}(u,v)$, $u, v \in [0,1]$, which has the considered parametric curves as its boundaries that is

$$\boldsymbol{r}(u,0) = \boldsymbol{p}_0(u)\,,\ \boldsymbol{r}(0,v) = \boldsymbol{q}_0(v)\,, \tag{6}$$

$$\boldsymbol{r}(u,1)=\boldsymbol{p}_1(u),\ \boldsymbol{r}(1,v)=\boldsymbol{q}_1(v). \quad (7)$$

In order to solve the problem determine the following parametric surface:

$$\boldsymbol{r}(u,v)=\boldsymbol{s}(u,v)-\tilde{\boldsymbol{r}}(u,v),\ u,v\in[0,1], \quad (8)$$

where

$$\boldsymbol{s}(u,v)=(1-w_{n+1}(v))\boldsymbol{p}_0(u)+w_{n+1}(v)\boldsymbol{p}_1(u)+ \quad (9)$$

$$+(1-w_{n+1}(u))\boldsymbol{q}_0(v)+w_{n+1}(u)\boldsymbol{q}_1(v)$$

$$\tilde{\boldsymbol{r}}(u,v)=(1-w_{n+1}(u))(1-w_{n+1}(v))\boldsymbol{r}_{0,0}+ \quad (10)$$

$$+w_{n+1}(u)(1-w_{n+1}(v))\boldsymbol{r}_{1,0}+$$

$$+(1-w_{n+1}(u))w_{n+1}(v)\boldsymbol{r}_{0,1}+w_{n+1}(u)w_{n+1}(v)\boldsymbol{r}_{1,1}$$

and the polynomials $w_{n+1}(u)$ are defined by Equation (1).

Substitution of the boundary values, which are defined by Equations (4-7), into the last equations yields that the parametric curves $\boldsymbol{p}_i(u)$ and $\boldsymbol{q}_i(v)$ are boundaries of the patch $\boldsymbol{r}(u,v)$. Besides it follows from boundary conditions (3) that the patch $\boldsymbol{r}(u,v)$ has the following partial derivatives at the corner points

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^m}(0,0)=(\boldsymbol{p}_0{}^{(m)}(u))(0),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial v^m}(0,0)=(\boldsymbol{q}_0{}^{(m)}(v))(0),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^m}(0,1)=(\boldsymbol{p}_1{}^{(m)}(u))(0),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial v^m}(0,1)=(\boldsymbol{q}_0{}^{(m)}(v))(1),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^m}(1,0)=(\boldsymbol{p}_0{}^{(m)}(u))(1),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial v^m}(1,0)=(\boldsymbol{q}_1{}^{(m)}(v))(0),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^m}(1,1)=(\boldsymbol{p}_1{}^{(m)}(u))(1),$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial v^m}(1,1)=(\boldsymbol{q}_1{}^{(m)}(v))(1)$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^r \partial v^s}(0,0)=0,\ \frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^r \partial v^s}(0,1)=0,$$

$$\frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^r \partial v^s}(1,0)=0,\ \frac{\partial^m \boldsymbol{r}(u,v)}{\partial u^r \partial v^s}(1,1)=0.$$

Detailed description of the presented approach is presented in the work [5].

4. MODELING BEZIER SURFACE PATCHES

Suppose that $\boldsymbol{p}_i(u)$, $u\in[0,1]$, $\boldsymbol{q}_i(v)$, $v\in[0,1]$, $i=1,2$, are Bézier curves that is

$$\boldsymbol{p}_i(u)=\sum_{k=0}^{2n+1} b_{2n+1,k}(u)\boldsymbol{p}_{i,k},\ u\in[0,1],$$

$$\boldsymbol{q}_i(v)=\sum_{l=0}^{2n+1} b_{2n+1,l}(v)\boldsymbol{q}_{i,l},\ v\in[0,1],$$

It is shown in the work [5] that in this case the patch, constructed by Equation (8), is a Bézier surface, which has the following representation:

$$\boldsymbol{r}(u,v)=$$

$$=\sum_{k=0}^{n} b_{2n+1,k}(u)\sum_{l=0}^{n} b_{2n+1,l}(v)(\boldsymbol{p}_{0,0,k}+\boldsymbol{q}_{0,0,l}-\boldsymbol{r}_{0,0})+$$

$$+\sum_{k=0}^{n} b_{2n+1,k}(u)\sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)(\boldsymbol{p}_{0,1,k}+\boldsymbol{q}_{0,0,l}-\boldsymbol{r}_{0,1})+$$

$$+\sum_{k=n+1}^{2n+1} b_{2n+1,k}(u)\sum_{l=0}^{n} b_{2n+1,l}(v)(\boldsymbol{p}_{0,0,k}+\boldsymbol{q}_{1,0,l}-\boldsymbol{r}_{1,0})+$$

$$+\sum_{k=n+1}^{2n+1} b_{2n+1,k}(u)\sum_{l=n+1}^{2n+1} b_{2n+1,l}(v)(\boldsymbol{p}_{0,1,k}+\boldsymbol{q}_{1,0,l}-\boldsymbol{r}_{1,1}).$$

It can be seen from these equations that the knot and control points of the Bézier surface $\boldsymbol{r}(u,v)$ can be arranged in a square block matrix

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{B}_{0,0} & \boldsymbol{B}_{0,1}\\ \boldsymbol{B}_{1,0} & \boldsymbol{B}_{1,1}\end{bmatrix}.$$

Here every internal block of the matrix corresponds to one of the surface equation term with corresponding corner point subscripts.

5. MODELING RATIONAL BEZIER SURFACE PATCHES

Now consider four rational Bézier curves $\boldsymbol{p}_i(u)$, $u\in[0,1]$, $\boldsymbol{q}_i(v)$, $v\in[0,1]$, $i=1,2$, in the affine space A

$$\boldsymbol{p}_i(u)=\frac{\sum_{k=0}^{2n+1} b_{2n+1,k}(u)w_{i,k}\boldsymbol{p}_{i,k}}{\sum_{k=0}^{2n+1} b_{2n+1,k}(u)w_{i,k}},\ u\in[0,1],$$

$$\boldsymbol{q}_i(v)=\frac{\sum_{l=0}^{2n+1} b_{2n+1,l}(v)w_{i,l}\boldsymbol{q}_{i,l}}{\sum_{l=0}^{2n+1} b_{2n+1,l}(v)w_{i,l}},\ v\in[0,1],$$

which have the same boundary points:

$$\boldsymbol{p}_0(0)=\boldsymbol{q}_0(0)=\boldsymbol{r}_{0,0},\ \ \boldsymbol{p}_0(1)=\boldsymbol{q}_1(0)=\boldsymbol{r}_{1,0},$$

$$\boldsymbol{p}_1(0)=\boldsymbol{q}_0(1)=\boldsymbol{r}_{0,1},\ \ \boldsymbol{p}_1(1)=\boldsymbol{q}_1(1)=\boldsymbol{r}_{1,1}.$$

The problem is to construct a rectangular patch $\boldsymbol{r}(u,v)$, $u,v\in[0,1]$, which has the considered parametric curves as its boundaries that is

$$\boldsymbol{r}(u,0)=\boldsymbol{p}_0(u),\ \boldsymbol{r}(0,v)=\boldsymbol{q}_0(v),$$

$$\boldsymbol{r}(u,1)=\boldsymbol{p}_1(u),\ \boldsymbol{r}(1,v)=\boldsymbol{q}_1(v).$$

In order to solve the problem, firstly represent the parametric curves $\boldsymbol{p}_i(u)$, $\boldsymbol{q}_i(v)$ using homogeneous coordinates as follows:

$$\boldsymbol{P}_i(u)=\sum_{k=0}^{2n+1} b_{2n+1,k}(u)\boldsymbol{P}_{i,k}\ ,\ u\in[0,1],$$

$$\boldsymbol{Q}_i(v)=\sum_{l=0}^{2n+1} b_{2n+1,l}(v)\boldsymbol{Q}_{i,l}\ ,\ v\in[0,1],$$

where

$$\boldsymbol{P}_{i,k}=(w_{i,k}\boldsymbol{p}_{i,k},w_{i,k}),$$

$$\boldsymbol{Q}_{i,l}=(w_{i,l}\boldsymbol{q}_{i,l},w_{i,l}).$$

Then, using Equation (8), construct the rational Bézier surface $\boldsymbol{R}_{i,j}(u,v)$.

$$\boldsymbol{R}(u,v)=\boldsymbol{S}(u,v)-\tilde{\boldsymbol{R}}(u,v),\ u,v\in[0,1],$$

Now make the inverse transition to Cartesian coordinates. It is obtained that

$$\boldsymbol{r}(u,v)=\frac{\boldsymbol{s}(u,v)-\tilde{\boldsymbol{r}}(u,v)}{s(u,v)-\tilde{r}(u,v)},\ u,v\in[0,1] \tag{11}$$

where the surfaces $\boldsymbol{s}(u,v)$ and $\tilde{\boldsymbol{r}}(u,v)$ are defined by means of Equations (9) and (10) respectively provided that

$$\boldsymbol{p}_i(u)=\sum_{k=0}^{2n+1} b_{2n+1,k}(u)w_{i,k}\boldsymbol{p}_{i,k}\ ,\ u\in[0,1],$$

$$\boldsymbol{q}_i(v)=\sum_{l=0}^{2n+1} b_{2n+1,l}(v)w_{i,l}\boldsymbol{q}_{i,l}\ ,\ v\in[0,1],$$

for $i=1,2$. That is the parametric curves $\boldsymbol{p}_i(u)$ and $\boldsymbol{q}_i(v)$ are numerators of the corresponding rational Bézier curves. Besides the corner points $\boldsymbol{r}_{i,j}$, $i,j=0,1$, which are used for the definition of the surface $\tilde{\boldsymbol{r}}(u,v)$, equipped with the corresponding weights $w_{i,j}$.

Further the denominator of the rational parametric surface $\boldsymbol{r}(u,v)$ is defined by means of the following real functions:

$$s(u,v)=(1-w_{n+1}(v))p_0(u)+w_{n+1}(v)p_1(u)+$$
$$+(1-w_{n+1}(u))q_0(v)+w_{n+1}(u)q_1(v)$$

and

$$\tilde{r}(u,v)=(1-w_{n+1}(u))(1-w_{n+1}(v))w_{0,0}+$$
$$+w_{n+1}(u)(1-w_{n+1}(v))w_{1,0}+$$
$$+(1-w_{n+1}(u))w_{n+1}(v)w_{0,1}+w_{n+1}(u)w_{n+1}(v)w_{1,1}$$

where

$$p_i(u)=\sum_{k=0}^{2n+1} b_{2n+1,k}(u)w_{i,k}\ ,\ u\in[0,1],$$

$$q_i(v)=\sum_{l=0}^{2n+1} b_{2n+1,l}(v)w_{i,l}q_{i,l}\ ,\ v\in[0,1],$$

for $i=1,2$. That is the real functions $p_i(u)$ and $q_i(v)$ can be considered as Bézier curves on weights of the corresponding points which form the rational Bézier curves. It can be seen that the inverse mapping from homogeneous to rational Bézier curves, which is described by Equality (11), is continuous provided that the denominator of Equation (11) is not equal to zero. This condition is fulfilled for the positive weights of points.

6. INTERPOLATION OF RECTANGULAR GRIDS

Consider a rectangular grid of points $\boldsymbol{p}_{ij}$, $i\in\{0,1,\ldots,m\}$, $j\in\{0,1,\ldots,n\}$, in an affine space *A*. The problem is to construct a C^n continuous surface which interpolates points of this grid. Using approximating polynomials $w_n(u)$, the problem can be solved as follows.

Firstly, it is necessary to construct mesh of parametric curves which will form borders of patches. The simplest way is to construct cubic Bézier spline curves which interpolate the grid of knot points. Detailed description of the approach can be found in the work [5]. Fig. 1 illustrates the approach. It is obvious that in this case all knot points are described using Cartesian coordinates. Therefore it can be supposed that unity weights are assigned to all knot points of the grid.

Now suppose that different weights are assigned to knot points of the grid. It is natural to assume that the weights are positive numbers. Then a mesh of parametric curves, which interpolates the grid of knot points, can be constructed using rational cubic Bézier curves [6].

Before discussing shape of surfaces it is helpful to consider shape of rational Bézier curves which from boundaries of the surface patches. Fig. 2 shows how the spline curve shape depends on weights of its knot points. Initial and end points of the spline curves have unit weights. The middle knot points of the spline curves have the following weights 2.0, 1.0, 0.15. It can be seen that the weights, which are greater than unity, makes shape of the curve sharper around the knot point. Conversely, the weights, which are smaller than unity, make shape of the curve flatter around the knot point. The same considerations can be applied to the surfaces.

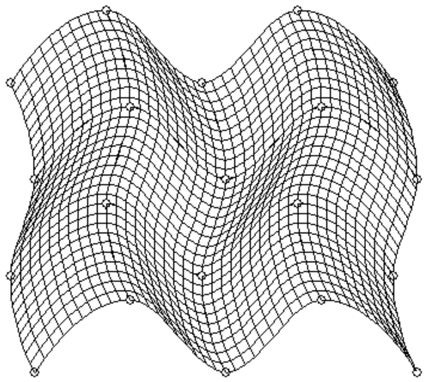

Fig. 1 – Unit weights are assigned to all knot points

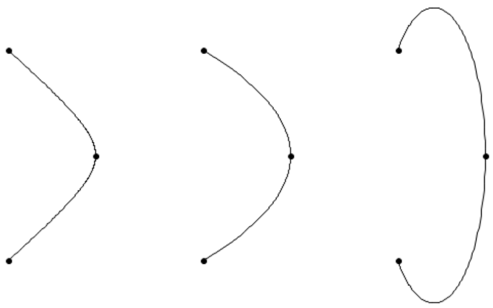

Fig. 2 – Shape of spline curves depending on weights

Fig. 3 and 4 illustrate how shape of the surface is changed when a weight of the internal knot point of the grid is changed. If the assigned weight is greater than unity than the surface shape becomes sharper around the knot point. Conversely, if the assigned weight is smaller than unity than the surface shape becomes flatter around the knot point.

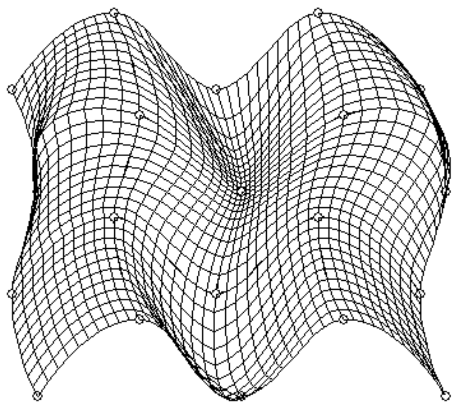

Fig. 3 – A weight 2.5 is assigned to the internal point of the surface

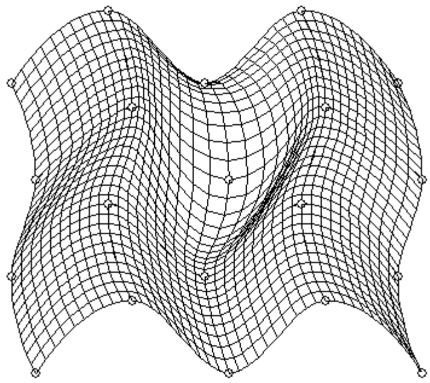

Fig. 4 – A weights of 0.5 is assigned to the internal point of the surface

It can be also noted that changing a weight of one knot point causes shape modification of patches, for which this point is not only a corner, but also of all patches which are adjacent to the mentioned ones. This is due to the algorithm which is used for construction of the curve mesh interpolating the grid. More detailed discussion of this issue can be found in the work [6].

The presented approach to surface modeling can be used when knot points of the surface are fixed, but shape of the surface, constructed by means of only knot points, doesn't meet the requirements.

Some other approaches to construction of interpolating surfaces over bivariate mesh of curves can be found in the works [7-10]. It should be noted that all these approaches are aimed at construction of C^1 continuous spline surfaces, while the presented one is more general because it can be used for modeling of surfaces with an arbitrary degree of continuity. More exhaustive bibliography on the topic with additional considerations can be found in the referenced works of the author.

At the end of this section it is necessary to note that constructed spline surface is also continuous along boundaries of patches as well as the spline surface constructed on the mesh of non-rational spline Bézier curves. It follows from continuities of spline Bézier curves in homogeneous coordinates and inverse mapping to Cartesian coordinates which is describe by Equation (11). But in this case computations of partial derivatives at knot points and along boundaries of patches are more sophisticated. More detailed consideration of the issue can be found in the work [6].

7. CONCLUSION

The paper presents a method to construct surfaces interpolating rectangular grids. The method is based on deformation of border curves using polynomials satisfying special boundary conditions. Analytical expressions for these polynomials by means of Bernstein polynomials are presented.

The distinguished feature of the presented approach to surface interpolation is that shape of the surface can be locally modified by assigning a weight to the knot point of the spline surface. This property can be useful when knot points of the surface are fixed, but shape of the surface, constructed by means of only knot points, doesn't meet the design requirements.

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